ON THE GENERAL INTEGRALS OF THE EQUATIONS OF MOTION OF A SOLID BODY THROUGH A FLUID

(K VOPROSU OB OBSHCHIKH INTEGRALAKH URAVNENII DVIZHENIIA TVERDOGO TELA V ZHIDKOSTI)

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As is known [1], the problem of the motion of a solid body immersed in an ideal homogeneous incompressible fluid filling an unbounded simply connected space in the absence of external forces and under the assumption that the motion of the fluid is nonvortical at the initial instant, reduces to the integration of a system of differential equations developed by Klebsch

$$\frac{dx_1}{dt} = x_2\frac{\partial T}{\partial y_3} - x_3\frac{\partial T}{\partial y_2}, \qquad \frac{dy_1}{dt} = x_2\frac{\partial T}{\partial x_3} - x_3\frac{\partial T}{\partial x_2} + y_2\frac{\partial T}{\partial y_3} - y_3\frac{\partial T}{\partial y_2} \qquad (1\,2\,3)$$

where T is a positively defined homogeneous quadratic form in the variables x_i, y_i $(i = 1, 2, 3)$ with constant coefficients.

As was determined by Kirchhoff, the Klebsch equations always have the three integrals

$$2T = M, \qquad x_1^2 + x_2^2 + x_3^2 = N, \qquad x_1y_1 + x_2y_2 + x_3y_3 = P$$

where M, N, P are arbitrary constants. Since the principle of the last Jacobi multiplier is applicable to the system of Klebsch equations the problem can be solved by finding one more (a fourth) integral which is not a consequence of the three Kirchhoff integrals.

There are five known cases (pointed out by Klebsch, Steklov and Liapunov) in which a fourth general integral exists [1 and 2]. Steklov [1] showed that in these and only in these cases the fourth integral represents a homogeneous second-degree polynomial in x_i, y_i $(i = 1, 2, 3)$.

All of the five classical cases of integrability have yet another general property: the fourth general integral in these cases can be written as

$$F(x_1, x_2, y_1, y_2) = \text{const}$$

(this can be achieved by constructing a linear combination of Kirchhoff integrals and the appropriate fourth integral).

We are now faced with the following question: are the Klebsch equations compatiable with the existence of a first integral of the form $F(x_1, x_2, y_1, y_2)$ = const in cases other than the classical ones? This question can be answered as follows without any additional assumptions about the form of the integral.

Let the body be referred to the second central coordinate system [1] in such a way that

$$2T = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}y_1^2 + b_{55}y_2^2 + b_{66}y_3^2 + 2b_{12}x_1x_2 + 2b_{23}x_2x_3 +$$
$$+ 2b_{13}x_1x_3 + 2b_{14}x_1y_1 + 2b_{25}x_2y_2 + 2b_{36}x_3y_3 + 2b_{24}(x_1y_2 + x_2y_1) + 2b_{35}(x_2y_3 + x_3y_2) +$$
$$+ 2b_{16}(x_3y_1 + x_1y_3)$$

If the body is arbitrary, then the parameters b_{ij} appearing in the expression for $2T$ are subject only to the condition that T be definitely positive; specifically, $b_{jj} > 0$.

Let $b_{23} = b_{13} = b_{35} = b_{16} = 0$. Introducing the notation

$$a = \frac{b_{44}}{b_{66}}, \quad \alpha = \frac{b_{11} - b_{33}}{b_{66}}, \quad \xi = \frac{b_{36} - b_{25}}{b_{66}}, \quad \varepsilon = \frac{b_{12}}{b_{66}}$$
$$b = \frac{b_{55}}{b_{66}}, \quad \beta = \frac{b_{33} - b_{22}}{b_{66}}, \quad \eta = \frac{b_{14} - b_{36}}{b_{66}}, \quad \zeta = \frac{b_{24}}{b_{66}}$$

and differentiating the relation $F(x_1, x_2, y_1, y_2) = \text{const}$ with respect to time, by virtue of the equations of motion we have

$$(-\zeta x_1 + \xi x_2 - by_2)\frac{\partial F}{\partial x_1} + (\eta x_1 + \zeta x_2 + ay_1)\frac{\partial F}{\partial x_2} + (-\varepsilon x_1 + \beta x_2 - \zeta y_1 + \xi y_2)\frac{\partial F}{\partial y_1} +$$
$$+ (\alpha x_1 + \varepsilon x_2 + \eta y_1 + \zeta y_2)\frac{\partial F}{\partial y_2} = 0$$
$$x_2\frac{\partial F}{\partial x_1} - x_1\frac{\partial F}{\partial x_2} + [-\zeta x_1 + \xi x_2 + (1-b)y_2]\frac{\partial F}{\partial y_1} + [\eta x_1 + \zeta x_2 + (a-1)y_1]\frac{\partial F}{\partial y_2} = 0$$

If we now convert to the new variables u, v, ρ, θ in such a way that the second of these equations becomes

$$\partial F / \partial \theta = 0$$

so that $F = F(u, v, \rho)$, the first equation breaks up into several linear equations whose investigation presents no fundamental difficulties. It should be noted, however, that the calculations involved are rather cumbersome. The form of the variable substitution depends on the sign and magnitude of the expression $(a-1)(b-1)$.

When $(a-1)(b-1) < 0$, only the second case of Klebsch and the Steklov case are possible; when $(a-1)(b-1) > 0$, only the first and second cases of Klebsch and the Steklov case are possible; when $(a-1)(b-1) = 0$ only the first and third cases of Klebsch and the Liapunov case are possible.

As regards the case

$$b_{23}^2 + b_{13}^2 + b_{35}^2 + b_{16}^2 \neq 0$$

it merely imposes additional equations on the function F provided the foregoing ones are valid, and is impossible in the five cases indicated.

Thus, a general integral of the form

$$F(x_1, x_2, y_1, y_2) = \text{const}$$

cannot apply to the Klebsch equations in cases other than the classical ones.

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